

SUPERCONVERGENCE ANALYSIS OF DG-FEM BASED ON THE POLYNOMIAL PRESERVING RECOVERY FOR HELMHOLTZ EQUATION WITH HIGH WAVE NUMBER

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Abstract. We study superconvergence property of the linear discontinuous Galerkin finite element method with the polynomial preserving recovery (PPR) and Richardson extrapolation for the two dimensional Helmholtz equation. The error estimate with explicit dependence on the wave number k , the penalty parameter μ and the mesh condition parameter α is derived. First, we prove that under the assumption $k(kh)^2 \leq C_0$ (h is the mesh size) and certain mesh condition, the estimate between the finite element solution and the linear interpolation of the exact solution is superconvergent under the $||| \cdot |||$ -seminorm. Second, we prove a superconvergence result for the recovered gradient by PPR. Furthermore, we estimate the error between the finite element gradient and recovered gradient, which motivate us to define the a posteriori error estimator. Finally, Some numerical examples are provided to confirm the theoretical results of superconvergence analysis. All theoretical findings are verified by numerical tests.

Key words. Helmholtz equation, large wave number, pollution errors, superconvergence, polynomial preserving recovery, discontinuous Galerkin finite element methods

AMS subject classifications. 65N12, 65N15, 65N30, 78A40

1. Introduction. This paper is devoted to superconvergence analysis of the linear Galerkin finite element method for the following Helmholtz problem:

$$(1.1) \quad -\Delta u - k^2 u = f \quad \text{in } \Omega,$$

$$(1.2) \quad \frac{\partial u}{\partial n} + \mathbf{i}ku = g \quad \text{on } \Gamma,$$

where $\Omega \in \mathbb{R}^2$ is a bounded polygon with boundary $\Gamma := \partial\Omega$, $\mathbf{i} = \sqrt{-1}$ denotes the imaginary unit and n denotes the unit outward normal to Γ . The above Helmholtz problem is an approximation of the acoustic scattering problem (with time dependence $e^{i\omega t}$) and k is known as the wave number. The Robin boundary condition (1.2) is known as the first order approximation of the radiation condition (cf. [16]). The Helmholtz problem (1.1)–(1.2) also arises in applications as a consequence of frequency domain treatment of attenuated scalar waves (cf. [13]).

It is well known that the finite element method of fixed order for the Helmholtz problem (1.1)–(1.2) at high frequencies ($k \gg 1$) is subject to the effect of pollution: the ratio of the error of the finite element solution to the error of the best approximation from the finite element space cannot be uniformly bounded with respect to k [2, 5, 4, 12, 19, 20, 21]. More precisely, the linear finite element method for a 2-D Helmholtz problem satisfies the following error estimate under the mesh constraint $k(kh)^2 \leq C_0$

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[37, 14]:

$$(1.3) \quad \|\nabla(u - u_h)\|_{L^2(\Omega)} \leq C_1 kh + C_2 k(kh)^2.$$

Here u_h is the linear finite element solution, h is the mesh size and $C_i, i = 1, 2$ are positive constants independent of k and h . It is easy to see that the order of the first term on the right hand side of (1.3) is the same to that of the interpolation error in H^1 -seminorm and it can dominate the error bound only if $k(kh)$ is small enough. However, the second term on the right-hand side of (1.3) dominates the estimate under other mesh conditions. For example, kh is fixed and k is large enough. The term $C_2 k(kh)^2$ is called the pollution error of the finite element solution.

Considerable efforts have been made in analysis of different numerical methods for the Helmholtz problem with large wave number in the literature. The readers are referred to [3, 13, 28] for asymptotic error estimates of general DG methods and [20, 21] for pre-asymptotic error estimates of a one-dimensional problem discretized on equidistant grid. For more pre-asymptotic error estimates, Please refer to [24, 25] and [8, 37] for classical finite element methods as well as interior penalty finite element methods. For other methods solving the Helmholtz problems, such as the interior penalty discontinuous Galerkin method or the source transfer domain decomposition method, one can read [23, 17, 18, 36, 15, 10].

In this work, we investigate the superconvergence property of the linear discontinuous Galerkin (DG) finite element method when being post-processed by our polynomial preserving recovery (PPR) for the Helmholtz problem. PPR was proposed by Zhang and Naga [35] in 2004 and has been successfully applied to finite element methods. COMSOL Multiphysics adopted PPR as a post-processing tool since 2008 [1]. [?] has applied the technique to the Helmholtz problem and prove its superconvergence property. In this paper, we generalize the technique over the DG finite element space and prove its superconvergence property. To learn more about PPR, readers are referred to [33, 32, 26, 29]. Some theoretical results about recovery techniques and recovery-type error estimators can be found in [6, 22, 34, 30, 31].

Our purpose of this paper is to prove the superconvergence error estimates for the linear discontinuous Galerkin finite element method and analyze the influence of the PPR technique on the pollution error. Note the superconvergence error estimates depend on the triangulation, the penalty parameter and the wave number under certain mesh condition. In order to prove the estimates, we first assume some mesh constraints, called “Condition α ”, and then redefine the PPR method on the discontinuous Galerkin finite element space. Finally, all the estimates motivate us to combine the PPR technique and the Richardson extrapolation to reduce the error further and define the a posterior error estimator.

The remainder of this paper is organized as follows: some notations, DG-FEM and the mesh constraints are introduced in section 2. In section 3, we prove the superconvergence between the interpolant and the finite element solution to the problem with Robin boundary (1.1)–(1.2). In section 4, we redefine the PPR technique over the linear DG finite element space and prove the superconvergence property of G_h in the Sobolev space H^3 and show the most important result, that is the error estimate of $G_h u_h$. Then we try to give the reason for the effect of G_h to the pollution error in section 5. Finally, we simulate a model problem by the linear DG-FEM, PPR method and the Richardson extrapolation in section 6. It is shown that the recovered gradient can be improved by the Richardson extrapolation further and the a posterior error estimator based on the PPR and Richardson extrapolation is exact asymptotically.

Throughout the paper, we assume that Ω is a strictly star-shaped domain. Here “strictly star-shaped” means that there exist a point $x_\Omega \in \Omega$ and a positive constant c_Ω depending only on Ω such that

$$(x - x_\Omega) \cdot n \geq c_\Omega \quad \forall x \in \Gamma.$$

2. Preliminaries. Throughout this paper, we assume that for any node point $z \in \mathcal{N}_h$, there exists at least one interior edge $e \in \mathcal{E}_h^I$ having z .

To introduce the method and simplify the analysis, we introduce some notation first. The standard Sobolev and Hilbert space, norm, and inner product notation are adopted (cf. [7, 11]). In particular, $(\cdot, \cdot)_Q$ and $\langle \cdot, \cdot \rangle_\Sigma$ for $\Sigma = \partial Q$ denote the L^2 -inner product on complex-valued $L^2(Q)$ and $L^2(\Sigma)$ spaces, respectively.

Let \mathcal{T}_h be a regular triangulation of the domain Ω . For any $\tau \in \mathcal{T}_h$, we denote by h_τ its diameter and by $|\tau|$ its area. Let $h = \max_{\tau \in \mathcal{T}_h} h_\tau$. Assume that $h_\tau \approx h$.

Let V_h be the approximation space of piecewise linear polynomials, that is,

$$V_h := \{v_h \in L^2(\Omega) : v_h|_\tau \in P_1(\tau) \quad \forall \tau \in \mathcal{T}_h\},$$

where $P_1(\tau)$ denotes the set of all polynomials on τ with degree ≤ 1 .

\mathcal{E}_h be the set of all edges of \mathcal{T}_h and \mathcal{N}_h be the set of all nodal points. Denote all the boundary edges by $\mathcal{E}_h^B := \{e \in \mathcal{E}_h : e \subset \Gamma\}$ and the interior edges by $\mathcal{E}_h^I := \mathcal{E}_h \setminus \mathcal{E}_h^B$. For each edge $e \in \mathcal{E}_h$, define $h_e := \text{diam}(e)$. For $e = \partial\tau \cap \tau' \in \mathcal{E}_h^I$, let \mathbf{n}_e be a unit normal vector to e . We assume that the normal vector \mathbf{n}_e is oriented from τ' to τ and define

$$[v]_e := v|_{\tau'} - v|_\tau, \quad \{v\} := \frac{1}{2}(v|_{\tau'} + v|_\tau).$$

We define the space $E := \prod_{\tau \in \mathcal{T}_h} H^1(\tau)$ and introduce the sesquilinear form $a_h(\cdot, \cdot)$ on $E \times E$ as follows:

$$\begin{aligned} a_h(u, v) &:= \sum_{K \in \mathcal{T}_h} (\nabla u, \nabla v)_K - \sum_{e \in \mathcal{E}_h^I} \left(\left\langle \left\{ \frac{\partial u}{\partial \mathbf{n}_e} \right\}, [v] \right\rangle_e + \left\langle [u], \left\{ \frac{\partial v}{\partial \mathbf{n}_e} \right\} \right\rangle_e \right) \\ &\quad + J_0(u, v) \\ J_0(u, v) &:= \sum_{e \in \mathcal{E}_h^I} \frac{\rho_{0,e}}{h_e^{1+\mu}} \langle [u], [v] \rangle_e, \end{aligned}$$

The linear DG method is defined as follows: find $u_h \in V_h$ such that

$$(2.1) \quad a_h(u_h, v_h) - k^2(u_h, v_h) + \mathbf{i}k \langle u_h, v_h \rangle = (f, v_h) + \langle g, v_h \rangle \quad \forall v_h \in V_h.$$

Remark 2.1 (a) Note that the method is the standard symmetric DG method (cf. [27]). So we have the proposition (cf. proposition 2.9 in [27]): $u \in H^2(\Omega)$ is the solution to (1.1)–(1.2) if and only if u satisfies the general DG variational formulation

$$a_h(u, v) - k^2(u, v) + \mathbf{i}k \langle u, v \rangle = (f, v) + \langle g, v \rangle \quad \forall v \in E.$$

(b) A similar DG method, called interior penalty discontinuous Galerkin method (IPDG), was introduced and analyzed by Feng, Wu and so on for the Helmholtz

problem (1.1)–(1.2). The reader is referred to [17, 18, 36, 15] for both asymptotic and preasymptotic error estimates.

The following two norms will be used in the forthcoming sections :

$$\begin{aligned} \|v\|_{1,h} &:= \left(\sum_{K \in \mathcal{T}_h} \|v\|_{L^2(K)}^2 + J_0(v, v) \right)^{1/2} \quad \forall v \in E, \\ \|v\|_{1,h} &:= \left(\|v\|_{1,h}^2 + k^2 \|v\|_0^2 \right)^{1/2} \quad \forall v \in E. \end{aligned}$$

We denote by $|\cdot|_{\mathcal{H}^1(\omega)}$ the broken semi- H^1 norm $\left(\sum_{\tau \in \omega} |\cdot|_{H^1(\tau)}^2 \right)^{1/2}$ where $\omega \subset \mathcal{T}_h$.

Throughout the paper C denotes a generic positive constant which is independent of h, k, f, g and the penalty parameters. We use the shorthand notation $A \lesssim B$ for $A \leq CB$ and assume $k \gg 1$ since we are considering high-frequency problems. We assume that the solution u to the problem (1.1)–(1.2) is H^3 -regular over Ω and the data g is H^2 -regular over Γ . Denote by

$$(2.2) \quad C_{u,g} = \sum_{j=1}^3 k^{-(j-1)} \|u\|_j + \sum_{j=1}^2 k^{-j} |g|_{H^j(\Gamma)}.$$

The function $C_{u,g}$ could be treated as a constant in this paper since $\|u\|_j$ is bounded by $\max(k^0, k^{j-1})$ when u is the solution to the Helmholtz problem (1.1)–(1.2). The reader is referred to [23, 24, 25] for the estimates of u .

Before estimating the errors, we state the coercivity and continuity properties for the sesquilinear form $a_h(\cdot, \cdot)$. Since they easily follow from the definitions 2.10–2.11 in [27], the details are omitted.

LEMMA 2.1. *For any $v, w \in E$, the sesquilinear form $a_h(\cdot, \cdot)$ satisfies*

$$|a_h(v, w)|, |a_h(w, v)| \lesssim \|w\|_{1,h} \|v\|_{1,h}.$$

In addition, there exists constant ρ such that if $\rho \leq \rho_{0,e}$,

$$\begin{aligned} \|v_h\|_{1,h}^2 &\lesssim a_h(v_h, v_h) \quad \forall v_h \in V_h, \\ \|v_h\|_{1,h}^2 &\lesssim a_h(v_h, v_h) + k^2 (v_h, v_h) \quad \forall v_h \in V_h. \end{aligned}$$

The following lemma shows the preasymptotic error estimates for the solution u to (1.1)–(1.2). The results can be derived by arguments same to those in [15, 36] and we omit the details to save space.

LEMMA 2.2. *Assume that u is the solution the problem (1.1)–(1.2) and u_h is the discrete solution of the scheme (2.1). Then there exists constant C_0 independent of k and h , such that if $k(kh)^2 \leq C_0$ then the following estimates hold:*

$$(2.3) \quad \|u - u_h\|_{1,h} \lesssim (kh + k(kh)^2) C_{u,g},$$

$$(2.4) \quad k\|u - u_h\|_0 \lesssim ((kh)^2 + k(kh)^2) C_{u,g}.$$

We begin with some definitions regarding meshes. For an interior edge $e \in \mathcal{E}_h^I$, we denote $\Omega_e = \tau_e \cup \tau'_e$, a patch formed by the two elements τ_e and τ'_e sharing e , see Figures 2.1-2.2. For any edge $e \in \mathcal{E}_h$ and an element τ with $e \subset \tau$, θ_e denotes

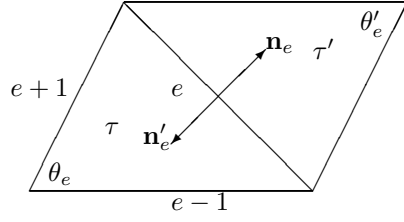
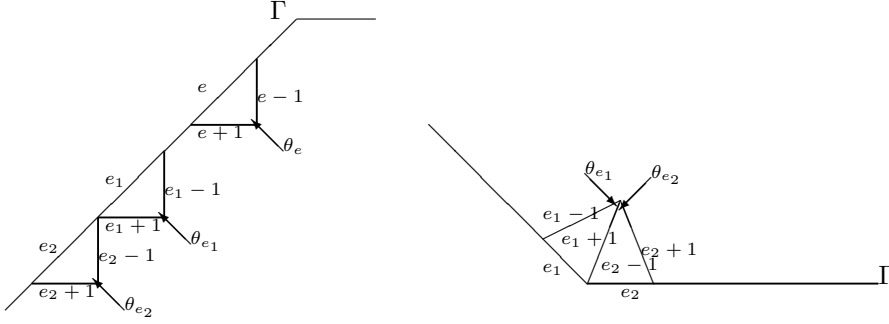
FIG. 2.1. Notation in the patch Ω_e .

FIG. 2.2. Notation in the boundary elements.

the angle opposite of the edge e in τ , \mathbf{t}_e denotes the unit tangent vector of e with counterclockwise orientation and \mathbf{n}_e , the unit outward normal vector of e , h_e, h_{e+1} , and h_{e-1} denote the lengths of the three edges of τ , respectively. Here the subscript $e+1$ or $e-1$ is for orientation. Note that all triangles in the triangulation are orientated counterclockwise, and the index $'$ is added for the corresponding quantities in τ' with $\mathbf{t}_e = -\mathbf{t}'_e$ and $\mathbf{n}_e = -\mathbf{n}'_e$ due to the orientation.

For any $e \in \mathcal{E}_h^I$ (cf. Figure 2.1), we say that Ω_e is an ε approximate parallelogram if the lengths of any two opposite edges differ by at most ε , that is,

$$|h_{e-1} - h'_{e-1}| + |h_{e+1} - h'_{e+1}| \leq \varepsilon.$$

For any $e \in \mathcal{E}_h^B$ (cf. Figure 2.2), we say that τ_e is an ε approximate isosceles triangle if the lengths of its two edges $e-1$ and $e+1$ differ by at most ε , that is,

$$|h_{e+1} - h_{e-1}| \leq \varepsilon.$$

DEFINITION 2.3. *The triangulation \mathcal{T}_h is said to satisfy mesh condition α if there exists a constant $\alpha \geq 0$ such that*

- (a) *the patch Ω_e is an $O(h^{1+\alpha})$ approximate parallelogram for any interior edge $e \in \mathcal{E}_h^I$;*
- (b) *the triangle τ_e is an $O(h^{1+\alpha})$ approximate isosceles triangle for any boundary edge $e \in \mathcal{E}_h^B$;*

Remark 2.1. The restriction (α) in Definition 2.3 is often used to prove the superconvergence property for problems with the Dirichlet boundary condition [9, 29]. Note that this restriction is technique and just for theoretical purpose. In fact, one superconvergence results still can be obtained under general meshes which do not satisfy the condition, such as Delaunay triangulation and Chevron pattern triangulation.

3. Superconvergence between the discontinuous finite element solution and linear interpolant. First we introduce a quadratic interpolant $\psi_Q = \Pi_Q \psi$ of ψ based on nodal values and moment conditions on edges,

$$(3.1) \quad (\Pi_Q \phi)(z) = \phi(z), \quad \int_e \Pi_Q \phi = \int_e \phi \quad \forall z \in \mathcal{N}_h, e \in \mathcal{E}_h.$$

The following fundamental identity for $v_h \in P_1(\tau)$ has been proved in [9]:

$$(3.2) \quad \int_\tau \nabla(\phi - \phi_I) \cdot \nabla v_h = \sum_{e \in \partial\tau} \left(\beta_e \int_e \frac{\partial^2 \phi_Q}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} + \gamma_e \int_e \frac{\partial^2 \phi_Q}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \right)$$

where

$$(3.3) \quad \beta_e = \frac{1}{12} \cot \theta_e (h_{e+1}^2 - h_{e-1}^2), \quad \gamma_e = \frac{1}{3} \cot \theta_e |\tau|,$$

and $\phi_I \in P_1(\tau)$ is the linear interpolant of ϕ on τ . The following lemma can be easily obtained [9, 29].

LEMMA 3.1. *Let \mathbf{m}_e denote \mathbf{t}_e or \mathbf{n}_e . Assume that \mathcal{T}_h satisfies the mesh condition α , then we have the following estimates:*

(a) *For any interior edge $e \in \mathcal{E}_h^I$,*

$$(3.4) \quad |\beta_e| + |\beta'_e| \lesssim h^2, \quad |\gamma_e| + |\gamma'_e| \lesssim h^2;$$

$$(3.5) \quad |\beta_e - \beta'_e| \lesssim h^{2+\alpha}, \quad |\gamma_e - \gamma'_e| \lesssim h^{2+\alpha}.$$

(b) *For two adjacent edges $e_1, e_2 \in \mathcal{E}_h^B$, that is $e_1 \cap e_2 \neq \emptyset$,*

$$(3.6) \quad |\beta_{e_1}| + |\beta_{e_2}| \lesssim h^{2+\alpha}, \quad |\gamma_{e_1}| + |\gamma_{e_2}| \lesssim h^2$$

$$(3.7) \quad |\gamma_{e_1} - \gamma_{e_2}| \lesssim h^{2+\alpha}$$

(c) *For any edge $e \in \mathcal{E}_h$, $e \subset \partial\tau_e$,*

$$(3.8) \quad \int_e \frac{\partial^2 \phi}{\partial \mathbf{t}_e \partial \mathbf{m}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \lesssim (\|\phi\|_{H^3(\tau_e)} + h^{-1} \|\phi\|_{H^2(\tau_e)}) |v_h|_{\mathcal{H}^1(\tau_e)};$$

$$(3.9) \quad \int_e \frac{\partial^2 (\phi - \phi_Q)}{\partial \mathbf{t}_e \partial \mathbf{m}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \lesssim |\phi|_{H^3(\tau_e)} |v_h|_{\mathcal{H}^1(\tau_e)}.$$

Proof. The inequalities (3.4)–(3.6) follow from the *mesh condition* α . From the condition (a) and (b) in Definition 2.3, we have for any $e_1, e_2 \in \mathcal{E}_h^B$ satisfying $e_1 \cap e_2 \neq \emptyset$ (cf. Figure 2.2),

$$\frac{|h_{e_1-1} h_{e_1+1} \cos \theta_{e_1} - h_{e_2-1} h_{e_2+1} \cos \theta_{e_2}|}{h} \lesssim h^{1+\alpha},$$

which implies (3.7).

Finally, the inequalities (3.8) and (3.9) follow from the trace theorem. \square

LEMMA 3.2. *Assume that \mathcal{T}_h satisfies the α approximation condition. Then for any $v_h \in V_h$,*

$$(3.10) \quad \left| \sum_{K \in \mathcal{T}_h} \int_K \nabla(u - u_I) \cdot \nabla v_h \right| \lesssim \left((kh)^2 + kh^{1+\alpha} + kh^{1+\mu/2} \right) \|v_h\|_{1,h} C_{u,g}.$$

Here u_I is the linear interpolant of u on Ω .

Proof. From (3.2), we have

$$\begin{aligned} \int_{\Omega} \nabla(u - u_I) \cdot \nabla v_h &= \sum_{\tau \in \mathcal{T}_h} \sum_{e \subset \partial \tau} \left(\beta_e \int_e \frac{\partial^2 u_Q}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} + \gamma_e \int_e \frac{\partial^2 u_Q}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \right) \\ &= I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \sum_{e \in \mathcal{E}_h^I} \left[\int_e \frac{\partial^2 u}{\partial \mathbf{t}_e^2} \left(\beta_e \frac{\partial v_h}{\partial \mathbf{t}_e} \Big|_{K_1} - \beta'_e \frac{\partial v_h}{\partial \mathbf{t}_e} \Big|_{K_2} \right) + \int_e \frac{\partial^2 u}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \left(\gamma_e \frac{\partial v_h}{\partial \mathbf{t}_e} \Big|_{K_1} - \gamma'_e \frac{\partial v_h}{\partial \mathbf{t}_e} \Big|_{K_2} \right) \right. \\ &\quad + \beta_e \int_e \frac{\partial^2(u_Q - u)}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} + \gamma_e \int_e \frac{\partial^2(u_Q - u)}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \\ &\quad \left. + \beta'_e \int_e \frac{\partial^2(u - u_Q)}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} + \gamma'_e \int_e \frac{\partial^2(u - u_Q)}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \right], \\ &= I_{1,1} + I_{1,2} + I_{1,3}, \\ I_2 &= \sum_{e \in \mathcal{E}_h^B} \left[\beta_e \int_e \frac{\partial^2 u}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} + \gamma_e \int_e \frac{\partial^2 u}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \right. \\ &\quad \left. + \beta_e \int_e \frac{\partial^2(u_Q - u)}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} + \gamma_e \int_e \frac{\partial^2(u_Q - u)}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \right]. \end{aligned}$$

First, $I_{1,2}$ and $I_{1,3}$ can be estimated by Lemma 3.1 and Hölder's inequality:

$$\begin{aligned} |I_{1,2} + I_{1,3}| &\lesssim \sum_{e \in \mathcal{E}_h^I} \left((h^{2+\alpha} + h^2) \|u\|_{H^3(\tau_e)} + h^{1+\alpha} \|u\|_{H^2(\tau_e)} \right) |v_h|_{\mathcal{H}^1(\tau_e)} \\ &\lesssim ((h^{2+\alpha} + h^2) \|u\|_3 + h^{1+\alpha} \|u\|_2) |v_h|_{\mathcal{H}^1} \\ &\lesssim ((kh)^2 + kh^{1+\alpha}) |v_h|_{\mathcal{H}^1} C_{u,g}. \end{aligned}$$

From Lemma 3.1 and the inverse inequality,

$$\begin{aligned} I_{1,1} &= \sum_{e \in \mathcal{E}_h^I} \left[(\beta_e - \beta'_e) \int_e \frac{\partial^2 u}{\partial \mathbf{t}_e^2} \frac{\partial v_h}{\partial \mathbf{t}_e} \Big|_{K_1} + (\gamma_e - \gamma'_e) \int_e \frac{\partial^2 u}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial \mathbf{t}_e} \Big|_{K_1} \right. \\ &\quad \left. + \beta'_e \int_e \frac{\partial^2 u}{\partial \mathbf{t}_e^2} \left[\frac{\partial v_h}{\partial \mathbf{t}_e} \right] + \gamma_e \int_e \frac{\partial^2 u}{\partial \mathbf{t}_e \partial \mathbf{n}_e} \left[\frac{\partial v_h}{\partial \mathbf{t}_e} \right] \right] \\ &\lesssim \sum_{e \in \mathcal{E}_h^I} \left[\left(h^{2+\alpha} \|u\|_{H^3(\tau_e)} + h^{1+\alpha} \|u\|_{H^2(\tau_e)} \right) |v_h|_{\mathcal{H}^1(\tau_e)} \right. \\ &\quad \left. + \left(h^{5/2} \|u\|_{H^3(\tau_e)} + h^{3/2} \|u\|_{H^2(\tau_e)} \right) \left(\sum_{e \in \mathcal{E}_h^I} \left\| \left[\frac{\partial v_h}{\partial \mathbf{t}_e} \right] \right\|_{L^2(e)}^2 \right)^{1/2} \right] \\ &\lesssim ((kh)^2 + kh^{1+\alpha} + kh^{1+\mu/2}) \|v_h\|_{1,h} C_{u,g}, \end{aligned}$$

where we have used the inequalities

$$\begin{aligned} \left(\sum_{e \in \mathcal{E}_h^I} \left\| \left[\frac{\partial v_h}{\partial \mathbf{t}_e} \right] \right\|_{L^2(e)}^2 \right)^{1/2} &= \left(\sum_{e \in \mathcal{E}_h^I} \left\| \frac{\partial [v_h]}{\partial \mathbf{t}_e} \right\|_{L^2(e)}^2 \right)^{1/2} \lesssim \left(\sum_{e \in \mathcal{E}_h^I} h_e^{-2} \|[v_h]\|_{L^2(e)}^2 \right)^{1/2} \\ &\lesssim h^{\mu/2-1/2} J_0(v_h, v_h)^{1/2} \leq h^{\mu/2-1/2} \|v_h\|_{1,h}. \end{aligned}$$

By combining the inequalities above, we get

$$(3.11) \quad |I_1| \lesssim \left((kh)^2 + kh^{1+\alpha} + kh^{1+\mu/2} \right) \|v_h\|_{1,h} C_{u,g}.$$

Then we turn to the estimate of I_2 . From (3.6) and (3.9),

$$(3.12) \quad \begin{aligned} & \sum_{e \in \mathcal{E}_h^B} \left[\beta_e \int_e \frac{\partial^2 u}{\partial t_e^2} \frac{\partial v_h}{\partial t_e} + \beta_e \int_e \frac{\partial^2 (u_Q - u)}{\partial t_e^2} \frac{\partial v_h}{\partial t_e} + \gamma_e \int_e \frac{\partial^2 (u_Q - u)}{\partial t_e \partial n_e} \frac{\partial v_h}{\partial t_e} \right] \\ & \lesssim \sum_{e \in \mathcal{E}_h^B} \left(h^{1+\alpha} \|u\|_{H^2(\tau_e)} + (h^{2+\alpha} + h^2) \|u\|_{H^3(\tau_e)} \right) |v_h|_{\mathcal{H}^1(\tau_e)} \\ & \lesssim (kh^{1+\alpha} + (kh)^2) |v_h|_{\mathcal{H}^1} C_{u,g}. \end{aligned}$$

Therefore, we only need to estimate the remaining terms of I_2 . Denote by z_i the nodes on Γ . Denote by $e_i, e_{i+1} \in \mathcal{E}_h^B$ sharing z_i with counterclockwise orientation (cf. Figure 2.2). Denote by $[\cdot]_{z_i} = (\cdot|_{e_{i+1}})(z_i) - (\cdot|_{e_i})(z_i)$ and $\{\cdot\}_{z_i} = (\cdot|_{e_i}(z_i) + \cdot|_{e_{i+1}}(z_i))/2$. Then we have

$$(3.13) \quad \begin{aligned} \sum_{e \in \mathcal{E}_h^B} \gamma_e \int_e \frac{\partial^2 u}{\partial t_e \partial n_e} \frac{\partial v_h}{\partial t_e} &= \sum_{z_i \in \Gamma \cap \mathcal{N}_h} [\gamma_e v_h]_{z_i} \frac{\partial^2 u}{\partial t_e \partial n_e}(z_i) - \sum_{e \in \mathcal{E}_h^B} \gamma_e \int_e \frac{\partial^3 u}{\partial t_e^2 \partial n_e} v_h, \\ &= I_{2,1} + I_{2,2} + I_{2,3}, \end{aligned}$$

where

$$\begin{aligned} I_{2,1} &= \sum_{z_i \in \Gamma \cap \mathcal{N}_h} [\gamma_e]_{z_i} \{v_h\}_{z_i} \frac{\partial^2 u}{\partial t_e \partial n_e}(z_i), \\ I_{2,2} &= \sum_{z_i \in \Gamma \cap \mathcal{N}_h} \{\gamma_e\}_{z_i} [v_h]_{z_i} \frac{\partial^2 u}{\partial t_e \partial n_e}(z_i), \\ I_{2,3} &= - \sum_{e \in \mathcal{E}_h^B} \gamma_e \int_e \frac{\partial^3 u}{\partial t_e^2 \partial n_e} v_h. \end{aligned}$$

Suppose that $w \in H^1([a, b])$ and denote by $h_{ab} = b - a$, then we have

$$(3.14) \quad \begin{aligned} w^2(b) &= \int_a^b \left(\frac{x-a}{b-a} w^2(x) \right)' dx = \frac{1}{b-a} \int_a^b w^2 + 2 \int_a^b \frac{x-a}{b-a} w w' \\ &\leq \frac{1}{h_{ab}} \|w\|_{L^2([a,b])}^2 + 2 |w|_{H^1([a,b])} \|w\|_{L^2([a,b])}, \end{aligned}$$

which implies

$$(3.15) \quad \begin{aligned} |I_{2,1}| &\leq \sum_{z_i \in \Gamma \cap \mathcal{N}_h} |[\gamma_e]_{z_i}| \left(\frac{1}{h_{e_i}} \left| \frac{\partial u}{\partial n_{e_i}} \right|_{H^1(e_i)}^2 + 2 \left| \frac{\partial u}{\partial n_{e_i}} \right|_{H^2(e_i)} \left| \frac{\partial u}{\partial n_{e_i}} \right|_{H^1(e_i)} \right)^{1/2} \\ &\quad \cdot \left(\frac{1}{h_{e_i}} \|v_h\|_{L^2(e_i \cap e_{i+1})}^2 + 2 |v_h|_{\mathcal{H}^1(e_i)} \|v_h\|_{L^2(e_i \cap e_{i+1})} \right)^{1/2} \\ &\lesssim \max_{z_i \in \Gamma \cap \mathcal{N}_h} |[\gamma_e]_{z_i}| \left(\frac{1}{h} \left| \frac{\partial u}{\partial n} \right|_{H^1(\Gamma)}^2 + \left| \frac{\partial u}{\partial n} \right|_{H^2(\Gamma)} \left| \frac{\partial u}{\partial n} \right|_{H^1(\Gamma)} \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
& \cdot \left(\frac{1}{h} \|v_h\|_{L^2(\Gamma)}^2 + |v_h|_{\mathcal{H}^1(\Gamma)} \|v_h\|_{L^2(\Gamma)} \right)^{1/2} \\
& \lesssim \frac{\max_{z_i \in \Gamma \cap \mathcal{N}_h} |\gamma_e|_{z_i}}{h} \left((|g|_{H^1(\Gamma)}^2 + k|u|_{H^1(\Gamma)}^2) + h(|g|_{H^2(\Gamma)} + k|u|_{H^2(\Gamma)}) \right) \\
& \quad \left(|g|_{H^1(\Gamma)} + k|u|_{H^1(\Gamma)} \right)^{1/2} \cdot \left(\|v_h\|_{L^2(\Gamma)}^2 + h|v_h|_{\mathcal{H}^1(\Gamma)} \|v_h\|_{L^2(\Gamma)} \right)^{1/2} \\
& \lesssim h^{1+\alpha} k^{3/2} \left(\|v_h\|_0 \|v_h\|_{\mathcal{H}^1} + h^{1/2} \|v_h\|_{\mathcal{H}^1}^{3/2} \|v_h\|_0^{1/2} \right)^{1/2} C_{u,g} \\
& \lesssim kh^{1+\alpha} \|v_h\|_{1,h} C_{u,g},
\end{aligned}$$

where we have used (3.7).

For any $z_i \in \mathcal{N}_h^B$, let $\tau_{i,1}, \tau_{i,2}, \dots, \tau_{i,n_i}$ be n_i triangles clockwise around z_i (cf. 3.1). If $n_i = 1$ for some $z_i \in \mathcal{N}_h^B$, it is easy to see that $[\gamma_e v_h]_{z_i}$ in (3.13) is equal to zero. Thus we assume that $n_i \geq 2$ for all $z_i \in \mathcal{N}_h^B$ for simplicity of presentation. Denote by $e_{ij} = \tau_{i,j} \cap \tau_{i,j+1}$ for $j = 1, 2, \dots, n_i - 1$. Clearly, e_{ij} are in \mathcal{E}_h^I and we have

$$| [v_h]_{z_i} | \leq \sum_{j=1}^{n_i-1} | [v_h]_{e_{ij}} | \leq \sqrt{n_i-1} \left(\sum_{j=1}^{n_i-1} | [v_h]_{e_{ij}} |^2 \right)^{1/2}.$$

It is well known that n_i can be bounded by a constant independent of the mesh size h for any regular triangulation. Therefore, we have

$$\begin{aligned}
(3.16) \quad |I_{2,2}| & \leq \sum_{z_i \in \mathcal{N}_h^B} |\{\gamma_e\}_{z_i}| \left(\frac{1}{h_{e_i}} \left| \frac{\partial u}{\partial n_{e_i}} \right|_{H^1(e_i)}^2 + 2 \left| \frac{\partial u}{\partial n_{e_i}} \right|_{H^2(e_i)} \left| \frac{\partial u}{\partial n_{e_i}} \right|_{H^1(e_i)} \right)^{1/2} \\
& \quad \cdot \left(\sum_{j=1}^{n_i-1} \frac{1}{h_{e_{ij}}} \| [v_h] \|_{L^2(e_{ij})}^2 + 2 | [v_h] |_{H^1(e_{ij})} \| [v_h] \|_{L^2(e_{ij})} \right)^{1/2} \\
& \lesssim \max_{z_i \in \mathcal{N}_h^B} |\{\gamma_e\}_{z_i}| \left(\frac{1}{h} \left| \frac{\partial u}{\partial n} \right|_{H^1(\Gamma)}^2 + \left| \frac{\partial u}{\partial n} \right|_{H^2(\Gamma)} \left| \frac{\partial u}{\partial n} \right|_{H^1(\Gamma)} \right)^{1/2} \\
& \quad \cdot \left(\frac{h^\mu}{\rho_0} \sum_{z_i \in \mathcal{N}_h^B} \sum_{j=1}^{n_i-1} \frac{\rho_{0,e_{ij}}}{h_{e_{ij}}^{1+\mu}} \| [v_h] \|_{L^2(e_{ij})}^2 \right)^{1/2} \\
& \lesssim h^{(\mu-1)/2} \max_{z_i \in \mathcal{N}_h^B} |\{\gamma_e\}_{z_i}| \left((|g|_{H^1(\Gamma)}^2 + k|u|_{H^1(\Gamma)}^2) + h(|g|_{H^2(\Gamma)} \right. \\
& \quad \left. + k|u|_{H^2(\Gamma)}) \cdot (|g|_{H^1(\Gamma)} + k|u|_{H^1(\Gamma)}) \right)^{1/2} J_0(v_h, v_h)^{1/2} \\
& \lesssim k^{3/2} h^{(3+\mu)/2} \|v_h\|_{1,h} C_{u,g}
\end{aligned}$$

where we have used (3.6).

On the other hand,

$$\begin{aligned}
(3.17) \quad |I_{2,3}| & \lesssim h^2 \left| \frac{\partial u}{\partial n} \right|_{H^2(\Gamma)} \|v_h\|_{L^2(\Gamma)} \\
& \lesssim h^2 (|g|_{H^2(\Gamma)} + k|u|_{H^2(\Gamma)}) \cdot \|v_h\|_0^{1/2} \|v_h\|_{\mathcal{H}^1}^{1/2}
\end{aligned}$$

$$\lesssim k^{3/2} h^2 \lll v_h \rrr_{1,h} C_{u,g}.$$

From (3.13)–(3.17),

$$(3.18) \quad \left| \sum_{e \in \mathcal{E}_h^B} \gamma_e \int_e \frac{\partial^2 u}{\partial t_e \partial n_e} \frac{\partial v_h}{\partial t_e} \right| \lesssim ((kh)^2 + kh^{1+\alpha} + k^{3/2} h^{(3+\mu)/2}) \lll v_h \rrr C_{u,g}.$$

Then the estimate of I_2 can be obtained from (3.12) and (3.18),

$$(3.19) \quad |I_2| \lesssim ((kh)^2 + kh^{1+\alpha} + k^{3/2} h^{(3+\mu)/2}) \lll v_h \rrr_{1,h} C_{u,g}.$$

Finally, by combining (3.11) and (3.19), we prove the lemma. \square

THEOREM 3.3. *Assume that \mathcal{T}_h satisfies the mesh condition α . u_h is the DG finite element solution of the scheme (2.1) and u_I is the linear interpolation of the solution u to (1.1)–(1.2). There exists a constant C_0 independent of k and h , such that if $k(kh)^2 \leq C_0$, we have*

$$(3.20) \quad \lll u_h - u_I \rrr_{1,h} \lesssim (kh^{1+\alpha} + kh^{1+\mu/2} + k(kh)^2) C_{u,g}.$$

Proof. For simplicity of presentation, we denote $v_h = u_h - u_I$. By Lemma 2.1 and the Galerkin orthogonality, we have

$$(3.21) \quad \begin{aligned} \lll u_h - u_I \rrr_{1,h}^2 &\lesssim a_h(u_h - u_I, v_h) + k^2(u_h - u_I, v_h) \\ &= \Re(a_h(u_h - u_I, v_h) - k^2(u_h - u_I, v_h) + \mathbf{i}k \langle u_h - u_I, v_h \rangle + 2k^2(u_h - u_I, v_h)) \\ &= \Re(a_h(u - u_I, v_h) - k^2(u - u_I, v_h) + \mathbf{i}k \langle u - u_I, v_h \rangle + 2k^2(u_h - u_I, v_h)) \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

It is well known that

$$(3.22) \quad |I_2| \leq k \|u - u_I\|_0 k \|v_h\| \lesssim kh^2 \|u\|_2 \lll v_h \rrr_{1,h}.$$

By the trace inequality, we have

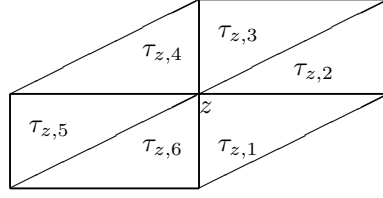
$$(3.23) \quad \begin{aligned} |I_3| &\leq k \|u - u_I\|_{L^2(\partial\Omega)} \|v_h\|_{L^2(\partial\Omega)} \\ &\lesssim kh^2 \|u\|_{H^2(\partial\Omega)} \|v_h\|_{L^2(\partial\Omega)} \\ &\lesssim k^{1/2} h^2 \|u\|_2^{1/2} \|u\|_3^{1/2} \lll v_h \rrr_{1,h} \\ &\lesssim (kh)^2 \lll v_h \rrr_{1,h} C_{u,g}. \end{aligned}$$

From Lemma 2.2, we know that there exists a constant C_0 independent on k and h , such that if $k(kh)^2 \leq C_0$, the following inequality holds,

$$k \|u - u_h\|_{L^2(\Omega)} \lesssim ((kh)^2 + k(kh)^2) C_{u,g},$$

which implies

$$(3.24) \quad \begin{aligned} |I_4| &\leq 2k \|u_h - u_I\|_0 \cdot k \|v_h\|_0 \\ &\leq 2(k \|u - u_h\|_0 + k \|u - u_I\|_0) \cdot k \|v_h\|_0 \\ &\lesssim ((kh)^2 + k(kh)^2) \lll v_h \rrr_{1,h} C_{u,g}. \end{aligned}$$

FIG. 3.1. n_z triangles sharing z with $n_z = 6$.

Next, we estimate I_1 . From the definition of $a_h(\cdot, \cdot)$ and the fact that u_I is continuous in Ω , we have

$$\begin{aligned} |I_1| &\leq \left| \sum_{\tau \in \mathcal{T}_h} \int_{\tau} \nabla(u - u_I) \cdot \nabla \bar{v}_h \right| + \left| \sum_{e \in \mathcal{E}_h^I} \left\langle \left\{ \frac{\partial(u - u_I)}{\partial \mathbf{n}_e} \right\}, [v_h] \right\rangle_e \right| \\ &= I_{1,1} + I_{1,2}. \end{aligned}$$

Then

$$\begin{aligned} (3.25) \quad I_{1,2} &\leq \sum_{e \in \mathcal{E}_h^I} \left\| \left\{ \frac{\partial(u - u_I)}{\partial \mathbf{n}_e} \right\} \right\|_{L^2(e)} \| [v_h] \|_{L^2(e)} \\ &\lesssim h^{1/2} \sum_{e \in \mathcal{E}_h^I} \|u\|_{H^2(\tau_e)} \| [v_h] \|_{L^2(e)} \\ &\lesssim h^{1+\mu/2} \|u\|_2 J_0(v_h, v_h)^{1/2} \\ &\lesssim kh^{1+\mu/2} \|v_h\|_{1,h} C_{u,g}. \end{aligned}$$

From Lemma 3.2 and (3.25),

$$(3.26) \quad |I_1| \lesssim \left((kh)^2 + kh^{1+\alpha} + kh^{1+\mu/2} \right) \|v_h\|_{1,h} C_{u,g}.$$

Therefore, by combining the equations (3.21)–(3.24) and (3.26), we have if $k(kh)^2 \leq C_0$, the following estimate holds:

$$\|u_h - u_I\|_{1,h}^2 \lesssim \left((kh)^2 + kh^{1+\alpha} + kh^{1+\mu/2} \right) \|v_h\|_{1,h} C_{u,g}.$$

This completes the proof. \square

In this paper, for any node point $z \in \mathcal{N}_h$, let n_z be the number of triangles associated with z and let $\tau_{z,1}, \tau_{z,2}, \dots, \tau_{z,n_z}$ be the elements having z such that they are counterclockwise around z as shown in Figure 3.1. $\tau_{z,j}$ means $\tau_{z, \text{mod}(j, n_z)}$ for the integer $j > n_z$.

LEMMA 3.4. Assume that \mathcal{T}_h satisfies the mesh condition α and u_h is the DG finite element solution of the scheme (2.1). For any $z \in \mathcal{N}_h$ and $\tau \in \mathcal{T}_h$ having the node point z , let $u_h(z, \tau)$ be $u_h|_{\tau}(z)$. There exists a constant C_0 independent of k and h , such that if $k(kh)^2 \leq C_0$, we have

$$\left(\sum_{z \in \mathcal{N}_h} \sum_{i=1}^{n_z} |u_h(z, \tau_{i+1}) - u_h(z, \tau_i)|^2 \right)^{1/2} \lesssim h^{\mu/2} \left(kh^{1+\alpha} + kh^{1+\mu/2} + k(kh)^2 \right) C_{u,g}.$$

Proof. For any $e \in \mathcal{E}_h^I$, let z_e, z'_e be its two endpoints and τ_e, τ'_e be two elements sharing e . Then from (3.14) and the discrete inverse inequalities, we have

$$\begin{aligned}
& \left(\sum_{z \in \mathcal{N}_h} \sum_{i=1}^{n_z} |u_h(z, \tau_{i+1}) - u_h(z, \tau_i)|^2 \right)^{1/2} \\
&= \left(\sum_{e \in \mathcal{E}_h^I} |u_h(z_e, \tau_e) - u_h(z_e, \tau'_e)|^2 + |u_h(z'_e, \tau_e) - u_h(z'_e, \tau'_e)|^2 \right)^{1/2} \\
&\lesssim \left(\sum_{e \in \mathcal{E}_h^I} \frac{1}{h_e} \| [u_h] \|_{L^2(e)}^2 + \| [u_h] \|_{L^2(e)} \| [u_h] \|_{H^1(e)} \right)^{1/2} \\
&\lesssim \left(\sum_{e \in \mathcal{E}_h^I} \frac{1}{h_e} \| [u_h] \|_{L^2(e)}^2 \right)^{1/2}.
\end{aligned}$$

From the fact that the linear interpolation u_I of the solution u to (1.1)–(1.2) is continuous, that is $[u_I]_e = 0 \ \forall e \in \mathcal{E}_H^I$, and Theorem 3.3,

$$\begin{aligned}
& \left(\sum_{e \in \mathcal{E}_h^I} \frac{1}{h_e} \| [u_h] \|_{L^2(e)}^2 \right)^{1/2} = \left(\sum_{e \in \mathcal{E}_h^I} \frac{1}{h_e} \| [u_h - u_I] \|_{L^2(e)}^2 \right)^{1/2} \\
&\lesssim h^{\mu/2} J_0(u_h - u_I, u_h - u_I)^{1/2} \lesssim h^{\mu/2} \| u_h - u_I \|_{1,h} \\
&\lesssim h^{\mu/2} \left((kh)^2 + kh^{1+\alpha} + kh^{1+\mu/2} \right) C_{u,g}.
\end{aligned}$$

This completes the proof. \square

4. The gradient recovery operator G_h and its superconvergence. In this section, we define a polynomial preserving recovery method for the discontinuous finite element space and derive the superconvergent error estimate.

We first recall a gradient recovery operator developed in 2004 for the continuous finite element methods, which is called polynomial preserving recovery (PPR). Let \tilde{V}_h be the approximation space of continuous piecewise linear polynomials over \mathcal{T}_h and let $\tilde{G}_h : C(\Omega) \mapsto \tilde{V}_h \times \tilde{V}_h$ be the gradient recovery operator. Given a node $z \in \mathcal{N}_h$, we select $n \geq 6$ sampling points $z_j \in \mathcal{N}_h$, $j = 1, 2, \dots, n$, in an element patch ω_z containing z (z is one of z_j) and fit a polynomial of degree 2, in the least squares sense, with values of $w \in C(\Omega)$ at those sampling points. First, we find $p_2 \in P_2(\omega_z)$ for some $w \in C(\Omega)$ such that

$$(4.1) \quad \sum_{j=1}^n (p_2 - w)^2(z_j) = \min_{q \in P_2} \sum_{j=1}^n (q - w)^2(z_j).$$

Here $P_2(\omega_z)$ is the well-known piecewise quadratic polynomial space defined on ω_z . The recovery gradient at z is then defined as

$$(4.2) \quad \tilde{G}_h w(z) = (\nabla p_2)(z).$$

We define a PPR operator G_h over the DG finite element space. For any $u_h \in V_h$, we define a continuous piecewise linear polynomial $\tilde{u}_h \in \tilde{V}_h$ by

$$\tilde{u}_h(z) = \lambda_1 u_h(z, \tau_{z,1}) + \lambda_2 u_h(z, \tau_{z,2}) + \dots + \lambda_{n_z} u_h(z, \tau_{z,n_z}) \quad \forall z \in \mathcal{N}_h,$$

where λ_j , $j = 1, 2, \dots, n_z$, are non-negative numbers satisfying $\lambda_1 + \lambda_2 + \dots + \lambda_{n_z} = 1$.

We define the gradient recovery operator G_h from V_h to $\tilde{V}_h \times \tilde{V}_h$ by

$$(4.3) \quad G_h u_h := \tilde{G}_h \tilde{u}_h.$$

Clearly, for any $v_h \in \tilde{V}_h$, $G_h v_h$ is completely equal to $\tilde{G}_h v_h$.

LEMMA 4.1. *For any node point $z \in \mathcal{N}_h$, let $\tau_{z,1}, \tau_{z,2}, \dots, \tau_{z,n_z}$ be the n_z elements counterclockwise around z as shown in Figure 3.1. The following inequality holds for any $u_h \in V_h$*

$$(4.4) \quad \|G_h u_h\|_0 \lesssim |u_h|_{\mathcal{H}^1(\mathcal{T}_h)} + \left(\sum_{z \in \mathcal{N}_h} \sum_{j=1}^{n_z-1} |u_h(z, \tau_{z,j+1}) - u_h(z, \tau_{z,j})|^2 \right)^{1/2}.$$

Proof. We have the property $\|\tilde{G}_h \tilde{v}_h\|_0 \lesssim \|\nabla \tilde{v}_h\|_0$ for $\tilde{v}_h \in \tilde{V}_h$ (cf. [26]), which implies

$$\|G_h u_h\|_0 = \|\tilde{G}_h \tilde{u}_h\|_0 \lesssim \|\nabla \tilde{u}_h\|_0 \lesssim |\tilde{u}_h - u_h|_{\mathcal{H}^1(\mathcal{T}_h)} + |u_h|_{\mathcal{H}^1(\mathcal{T}_h)}.$$

For any $\tau \in \mathcal{T}_h$, let $\phi_{\tau,1}, \phi_{\tau,2}$ and $\phi_{\tau,3}$ be its node bases and let $z_{\tau,1}, z_{\tau,2}, z_{\tau,3} \in \mathcal{N}_h$ be its three vertices satisfying $\phi_{\tau,i}(z_{\tau,j}) = \delta(i-j)$. We have

$$\begin{aligned} |\tilde{u}_h - u_h|_{\mathcal{H}^1(\mathcal{T}_h)} &= \left(\sum_{\tau \in \mathcal{T}_h} \|\nabla(\tilde{u}_h - u_h)\|_{L^2(\tau)}^2 \right)^{1/2} \\ &= \left(\sum_{\tau \in \mathcal{T}_h} \int_{\tau} \left| \sum_{j=1}^3 (\tilde{u}_h(z_{\tau,j}) - u_h(z_{\tau,j}, \tau)) \partial_x \phi_{\tau,j} \right|^2 \right. \\ &\quad \left. + \int_{\tau} \left| \sum_{j=1}^3 (\tilde{u}_h(z_{\tau,j}) - u_h(z_{\tau,j}, \tau)) \partial_y \phi_{\tau,j} \right|^2 \right)^{1/2} \\ &\leq \left(\sum_{\tau \in \mathcal{T}_h} \left(\sum_{j=1}^3 |\tilde{u}_h(z_{\tau,j}) - u_h(z_{\tau,j}, \tau)|^2 \right) \cdot \left(\sum_{j=1}^3 \int_{\tau} |\partial_x \phi_{\tau,j}|^2 + |\partial_y \phi_{\tau,j}|^2 \right) \right)^{1/2}. \end{aligned}$$

Since \mathcal{T}_h is a uniform regular triangulation, it is well known that $\sum_{j=1}^3 \int_{\tau} |\partial_x \phi_{\tau,j}|^2 + |\partial_y \phi_{\tau,j}|^2$ can be bounded by some constant C independent of the mesh size h and the triangle τ for any $\tau \in \mathcal{T}_h$. Therefore, we have

$$\begin{aligned} |\tilde{u}_h - u_h|_{\mathcal{H}^1(\mathcal{T}_h)} &\lesssim \left(\sum_{\tau \in \mathcal{T}_h} \left(\sum_{j=1}^3 |\tilde{u}_h(z_{\tau,j}) - u_h(z_{\tau,j}, \tau)|^2 \right) \right)^{1/2} \\ &\lesssim \left(\sum_{z \in \mathcal{N}_h} \sum_{j=1}^{n_z} |\tilde{u}_h(z) - u_h(z, \tau_{z,j})|^2 \right)^{1/2}. \end{aligned}$$

From the definition of \tilde{u}_h , we have for any $z \in \mathcal{N}_h$,

$$\begin{aligned}
\sum_{j=1}^{n_z} |\tilde{u}_h(z) - u_h(z, \tau_{z,j})|^2 &= \sum_{j=1}^{n_z} \left| \sum_{i \neq j} \lambda_i (u_h(z, \tau_{z,i}) - u_h(z, \tau_{z,j})) \right|^2 \\
&\leq \sum_{j=1}^{n_z} \left(\sum_{i \neq j} \lambda_i |u_h(z, \tau_{z,i}) - u_h(z, \tau_{z,j})| \right)^2 \\
&\leq \sum_{j=1}^{n_z} (n_z - 1) \sum_{i \neq j} \lambda_i^2 |u_h(z, \tau_{z,i}) - u_h(z, \tau_{z,j})|^2 \\
&\leq (n_z - 1) \sum_{j=1}^{n_z-1} \sum_{i=j+1}^{n_z} (\lambda_i^2 + \lambda_j^2) |u_h(z, \tau_{z,i}) - u_h(z, \tau_{z,j})|^2 \\
&\leq (n_z - 1) \sum_{j=1}^{n_z-1} \sum_{i=j+1}^{n_z} (\lambda_i^2 + \lambda_j^2) \left(\sum_{t=j}^{i-1} |u_h(z, \tau_{z,t+1}) - u_h(z, \tau_{z,t})| \right)^2 \\
&\leq (n_z - 1)^2 \sum_{j=1}^{n_z-1} \sum_{i=j+1}^{n_z} \sum_{t=i}^{j-1} (\lambda_i^2 + \lambda_j^2) |u_h(z, \tau_{z,t+1}) - u_h(z, \tau_{z,t})|^2 \\
&\leq (n_z - 1)^2 \sum_{t=1}^{n_z-1} \left((n_z - t) \sum_{i=1}^t \lambda_i^2 + t \sum_{i=t+1}^{n_z} \lambda_i^2 \right) |u_h(z, \tau_{z,t+1}) - u_h(z, \tau_{z,t})|^2 \\
&\leq (n_z - 1)^3 \sum_{t=1}^{n_z-1} |u_h(z, \tau_{z,t+1}) - u_h(z, \tau_{z,t})|^2.
\end{aligned}$$

Since n_z can be bounded by a constant independent of h and the vertex z , we complete the proof. \square

LEMMA 4.2. *For any element $\tau \in \mathcal{T}_h$ and any function $\phi \in H^3(\tilde{\tau})$,*

$$(4.5) \quad \|G_h \phi_I - \nabla \phi\|_{L^2(\tau)} \lesssim h^2 \|\phi\|_{H^3(\tilde{\tau})},$$

where $\tilde{\tau} = \bigcup \{\omega_z : z \in \mathcal{N}_h \cap \tau\}$ and ϕ_I is the linear interpolant of ϕ .

Proof. The proof is completed by the fact that $G_h u_I = \tilde{G}_h u_I$ and Lemma 4.1 in ?? . \square

Since u_I is continuous, that is

$$\tilde{u}_I(z) - u_I(z, \tau_{z,j}) = 0 \quad \forall z \in \mathcal{N}_h \text{ and } j = 1, 2, \dots, n_z,$$

we have

$$\begin{aligned}
(4.6) \quad \|G_h u_h - \nabla u\|_0 &\leq \|G_h(u_h - u_I)\|_0 + \|G_h u_I - \nabla u\|_0 \\
&\lesssim |u_h - u_I|_{\mathcal{H}^1(\mathcal{T}_h)} + \|G_h u_I - \nabla u\|_0 + \left(\sum_{z \in \mathcal{N}_h} \sum_{j=1}^{n_z} |u_h(z) - u_h(z, \tau_{z,j})|^2 \right)^{1/2}.
\end{aligned}$$

Then by combining Lemmas 3.2–4.2 and the inequality (4.6), we have the following theorem which is our main result in the paper.

THEOREM 4.3. *Let u and u_h be the solutions to (1.1)–(1.2) and the discrete solution, respectively. Assume that \mathcal{T}_h satisfies the mesh condition α . Then there exists a constant C_0 independent of k and h such that if $k(kh)^2 \leq C_0$,*

$$(4.7) \quad \|G_h u_h - \nabla u\|_0 \lesssim \left(kh^{1+\alpha} + kh^{1+\mu/2} + k(kh)^2 \right) C_{u,g}.$$

5. The influence of the operator G_h to the pollution error. In this section, we estimate the error between $G_h u_h$ and ∇u_h , which motivate us to combine the Richardson extrapolation and the ppr technique to reduce the numerical errors, and define the a posteriori estimator in Section 6.

First, we define an elliptic projection from V to V_h : find $u_h^+ \in V_h$ such that

$$(5.1) \quad a_h(u_h^+, v_h) + \mathbf{i}k \langle u_h^+, v_h \rangle = a_h(u, v_h) + \mathbf{i}k \langle u, v_h \rangle \quad \forall v_h \in V_h.$$

In other words, the elliptic projection u_h^+ of u is the finite element approximation to the solution of the following (complex-valued) Poisson problem:

$$(5.2) \quad -\Delta u = F \quad \text{in } \Omega,$$

$$(5.3) \quad \frac{\partial u}{\partial n} + \mathbf{i}k u = g \quad \text{on } \Gamma,$$

for some given function F which are determined by u . This kind of elliptic projection is often used to study some properties, such as stability and convergence, of the FEM for the Helmholtz problem. Readers are referred to [37, 36, 15, 14].

LEMMA 5.1. *Assume that u is H^2 -regular. u_h^+ is its elliptic projection defined by (5.1). There hold the following estimates:*

$$\begin{aligned} \|u - u_h^+\|_{1,h} &\lesssim \inf_{v_h \in V_h} \|u - v_h\|_{1,h} \\ \|u - u_h^+\|_0 &\lesssim h \inf_{v_h \in V_h} \|u - v_h\|_{1,h}. \end{aligned}$$

Proof. From Lemma 3.5 in [36], we know that

$$\begin{aligned} \|u - u_h^+\|_{1,h} &\lesssim \inf_{v_h \in V_h} \left(\|u - v_h\|_1^2 + k \|u - v_h\|_{L^2(\Gamma)}^2 \right)^{1/2}, \\ \|u - u_h^+\|_0 &\lesssim h \inf_{v_h \in V_h} \left(\|u - v_h\|_1^2 + k \|u - v_h\|_{L^2(\Gamma)}^2 \right)^{1/2}. \end{aligned}$$

Then the estimates follow from

$$\begin{aligned} k \|u - v_h\|_{L^2(\Gamma)}^2 &\lesssim k \|u - v_h\|_0 \|u - v_h\|_{\mathcal{H}^1(\Omega)} \\ &\lesssim k^2 \|u - v_h\|_0^2 + \|u - v_h\|_{\mathcal{H}^1(\Omega)}^2 \lesssim \|u - v_h\|_{1,h}^2. \end{aligned}$$

□

LEMMA 5.2. *Assume that u is the exact solution to (1.1)–(1.2) and u_h^+ is its elliptic projection defined by (5.1). Assume that \mathcal{T}_h satisfies the mesh condition α . We have*

$$(5.4) \quad \|\nabla u_h^+ - \nabla u_I\|_{1,h} \lesssim (kh^{1+\alpha} + kh^{1+\mu/2} + (kh)^2) C_{u,g}.$$

Proof. Denote $v_h = u_h^+ - u_I$. By the Galerkin orthogonality,

$$\begin{aligned} \|\nabla u_h^+ - \nabla u_I\|_{1,h}^2 &\lesssim \Re(a_h(u_h^+ - u_I, v_h) + \mathbf{i}k \langle u_h^+ - u_I, v_h \rangle) \\ &\quad + k^2 (u_h^+ - u_I, v_h) \\ &\lesssim \Re(a_h(u - u_I, v_h) + \mathbf{i}k \langle u - u_I, v_h \rangle) + k^2 (u_h^+ - u_I, v_h) \\ &\lesssim |a_h(u - u_I, v_h)| + |k \langle u - u_I, v_h \rangle| + k \|u_h^+ - u_I\|_0 \cdot k \|v_h\|_0. \end{aligned}$$

By some arguments same to those in 3.3, it is obtained that

$$|a_h(u - u_I, v_h)| \lesssim \left((kh)^2 + kh^{1+\alpha} + kh^{1+\mu/2} \right) \|v_h\|_{1,h} C_{u,g}$$

and

$$|k \langle u - u_I, v_h \rangle| \lesssim (kh)^2 \|v_h\|_{1,h} C_{u,g}.$$

From Lemma 5.1, we have

$$\begin{aligned} k \|u_h^+ - u_I\|_0 \cdot k \|v_h\|_0 &\leq (k \|u_h^+ - u\|_0 + k \|u - u_I\|_0) \|v_h\|_{1,h} \\ &\lesssim (kh)^2 (1 + \sqrt{kh}) \|v_h\|_{1,h} C_{u,g}. \end{aligned}$$

By combining the inequalities above, we complete the proof. \square

THEOREM 5.3. *Let u and u_h^+ be the solution to (1.1)–(1.2) and the elliptic projection defined by (5.1), respectively. Assume that the mesh condition α is satisfied. Then the following error estimate holds:*

$$(5.5) \quad \|G_h u_h^+ - \nabla u\|_0 \lesssim (kh^{1+\alpha} + kh^{1+\mu/2} + (kh)^2) C_{u,g}.$$

THEOREM 5.4. *Let u and u_h^+ be the solution to (1.1)–(1.2) and the elliptic projection defined by (5.1), respectively. Assume that the mesh condition α is satisfied. We have*

$$(5.6) \quad \left(\sum_{\tau \in \mathcal{T}_h} \|G_h u_h - \nabla u_h\|_{L^2(\tau)}^2 \right)^{1/2} \lesssim (kh + k(kh)^3) C_{u,g}.$$

Proof. Denote by $\theta_h = u_h - u_h^+$. u_h can be written as $u_h = u_h^+ + \theta_h$, where u_h^+ is the elliptic projection of u defined by (5.1). We have

$$\begin{aligned} (5.7) \quad &\left(\sum_{\tau \in \mathcal{T}_h} \|G_h u_h - \nabla u_h\|_{L^2(\tau)}^2 \right)^{1/2} \\ &= \left(\sum_{\tau \in \mathcal{T}_h} \|G_h(u_h^+ + \theta_h) - \nabla(u_h^+ + \theta_h)\|_{L^2(\tau)}^2 \right)^{1/2} \\ &\leq \left(\sum_{\tau \in \mathcal{T}_h} \|G_h u_h^+ - \nabla u_h^+\|_{L^2(\tau)}^2 \right)^{1/2} + \left(\sum_{\tau \in \mathcal{T}_h} \|G_h \theta_h - \nabla \theta_h\|_{L^2(\tau)}^2 \right)^{1/2}. \end{aligned}$$

From Lemma 5.1 and Theorem 5.3,

$$\begin{aligned} (5.8) \quad &\left(\sum_{\tau \in \mathcal{T}_h} \|G_h u_h^+ - \nabla u_h^+\|_{L^2(\tau)}^2 \right)^{1/2} \\ &\leq \|G_h u_h^+ - \nabla u\|_0 + |u - u_h^+|_{\mathcal{H}^1} \\ &\lesssim (kh^{1+\alpha} + kh^{1+\mu/2} + (kh)^2) C_{u,g} + kh C_{u,g} \\ &\lesssim (kh + (kh)^2) C_{u,g}. \end{aligned}$$

From (2.1) and (5.1), we have that θ_h satisfies the following equation:

$$(5.9) \quad a_h(\theta_h, v_h) + \mathbf{i}k \langle \theta_h, v_h \rangle = -k^2(u - u_h, v_h).$$

Therefor, θ_h can be understood as the numerical approximation to the following Poisson problem with Robin boundary:

$$\begin{aligned} -\Delta\theta &= -k^2(u - u_h) \quad \text{in } \Omega, \\ \frac{\partial\theta}{\partial n} + \mathbf{i}k\theta &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Therefore,

$$(5.10) \quad \left(\sum_{\tau \in \mathcal{T}_h} \|G_h\theta_h - \nabla\theta_h\|_{L^2(\tau)}^2 \right)^{1/2} \leq \|G_h\theta_h - \nabla\theta\|_0 + |\theta - \theta_h|_{\mathcal{H}^1} \\ \lesssim h \|\theta\|_2 \lesssim k^2 h \|u - u_h\|_0 \\ \lesssim k^2 h (kh^2 + k^2 h^2) C_{u,g} \\ \lesssim ((kh)^3 + k(kh)^3) C_{u,g}.$$

The proof is completed by combining (5.7)–(5.10). \square

6. Numerical Tests. In this section, some numerical tests are implemented in order to demonstrate our theoretical results. We simulate the Helmholtz problem (1.1)–(1.2) where the source data f and g is so chosen that the exact solution is

$$u = \frac{\cos(kr)}{r} - \frac{\cos k + \mathbf{i} \sin k}{k(J_0(k) + \mathbf{i}J_1(k))} J_0(kr)$$

in polar coordinates, where $J_0(z)$ is a Bessel function of the first kind, and $\Omega = [0.5, 1.5] \times [0.5, 1.5]$. In this section, let $\gamma_0 = 5$ and we set $\lambda_1 = 1, \lambda_2 = 0, \dots, \lambda_{n_\varepsilon} = 0$ in the definition of G_h .

We first simulate the problem over the regular pattern uniform triangulation and denote by \mathcal{T}_N the triangulation consisting of $2N^2$ triangles of size h which is equivalent to $1/N$.

From Theorem 3.3, there exists the estimate that if $k(kh)^2 \leq C_0$,

$$\|u_h - u_I\|_{1,h} \lesssim kh^{1+\mu/2} + k(kh)^2.$$

The left graphs in Figure 6.1–Figure 6.3 show the numerical errors $\|u_h - u_I\|_{1,h}$ with penalty parameters $\mu = 0, 1, 2$ for $k = 5, 10, 50$ and 100 , respectively. The right graphs in Figure 6.1–Figure 6.3 show the convergence orders of the errors $\|u_h - u_I\|_{1,h}$ shown in the left graphs, respectively. As we expected, $\|u_h - u_I\|_{1,h}$ decays at the rates of $O(h^1)$, $O(h^{3/2})$ and $O(h^2)$ for the small wave numbers $k = 5, 10$, respectively. However, we can see that for the large wave number $k = 50, 100$, $\|u_h - u_I\|_{1,h}$ does not converge at first, then begins to decay at the rates which are greater than $O(h^{1+\mu/2})$ when N is large enough, which implies the existence of the constraint $k(kh)^2 \leq C_0$ and the so-called pollution error $k(kh)^2$.

Figure 6.4–Figure ?? show the numerical errors $\|G_h u_h - \nabla u\|_0$ in left graphs and the convergence order in right graphs for $k = 5, 10, 50, 100$ with penalty parameters $\mu = 0, 1, 2$, respectively. Clearly, the recovered gradients super-converge at the rate greater than $O(h^{1+\mu/2})$. Therefore, whether the estimate (4.7) is sharp with respect to μ is still open. The constraint $k(kh)^2 \leq C_0$ and the so-called pollution error $k(kh)^2$ can also be observed.

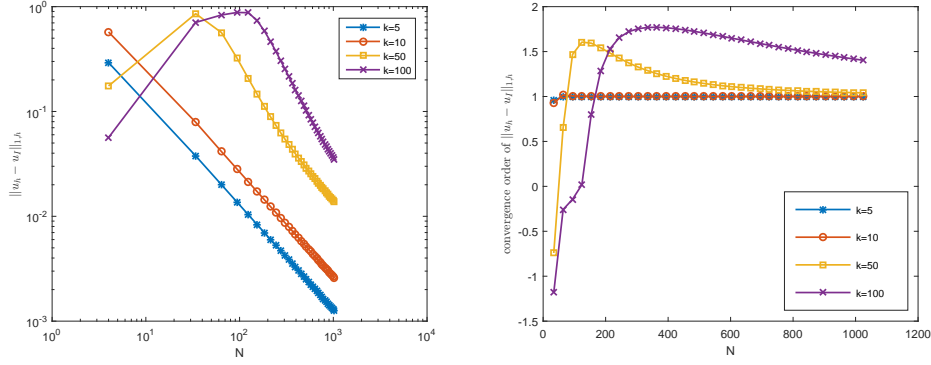


FIG. 6.1. $\|u_h - u_I\|_{1,h}$ (left) and the convergence order of $\|u_h - u_I\|_{1,h}$ (right) for $k = 5, 10, 50, 100$, where u_h is the numerical solution over the regular pattern uniform triangulation \mathcal{T}_N with $\mu = 0$.

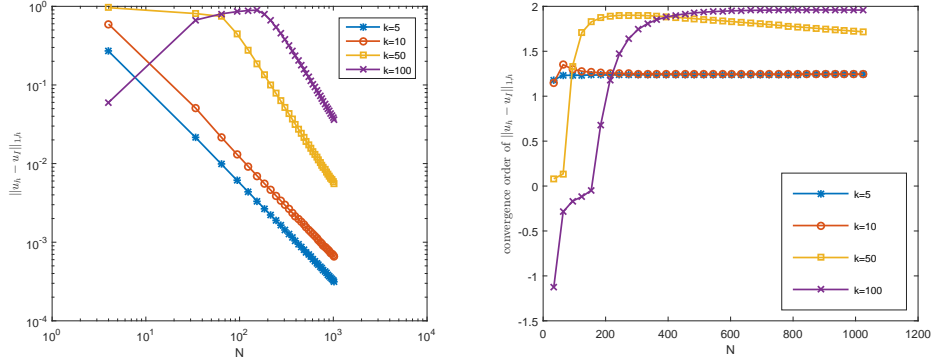


FIG. 6.2. $\|u_h - u_I\|_{1,h}$ (left) and the convergence order of $\|u_h - u_I\|_{1,h}$ (right) for $k = 5, 10, 50, 100$, where u_h is the numerical solution over the regular pattern uniform triangulation \mathcal{T}_N with $\mu = 1/2$.

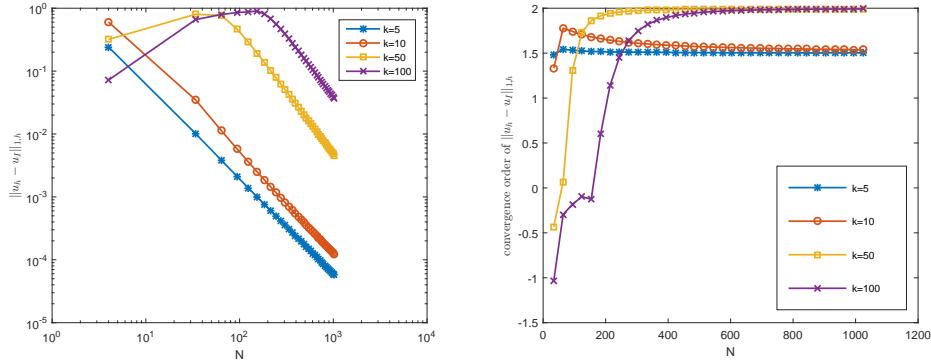


FIG. 6.3. $\|u_h - u_I\|_{1,h}$ (left) and the convergence order of $\|u_h - u_I\|_{1,h}$ (right) for $k = 5, 10, 50, 100$, where u_h is the numerical solution over the regular pattern uniform triangulation \mathcal{T}_N with $\mu = 1$.

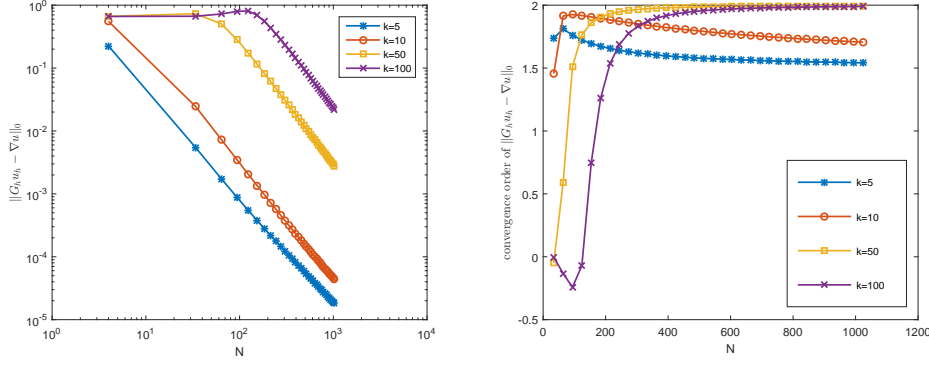


FIG. 6.4. $\|G_h u_h - \nabla u\|_0$ (left) and the convergence order of $\|G_h u_h - \nabla u\|_0$ (right) for $k = 5, 10, 50, 100$, where u_h is the numerical solution over the regular pattern uniform triangulation \mathcal{T}_N with $\mu = 0$.

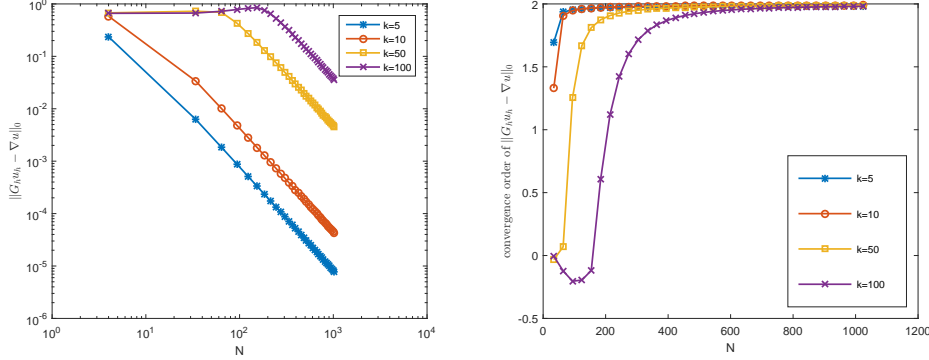


FIG. 6.5. $\|G_h u_h - \nabla u\|_0$ (left) and the convergence order of $\|G_h u_h - \nabla u\|_0$ (right) for $k = 5, 10, 50, 100$, where u_h is the numerical solution over the regular pattern uniform triangulation \mathcal{T}_N with $\mu = 1/2$.

| m | k=10 | | | k=50 | | |
|------|------------|------------|------------|------------|------------|------------|
| | E_1 | E_2 | E_3 | E_1 | E_2 | E_3 |
| 4 | 6.9541e-01 | 6.5276e-01 | | 8.9283e-01 | 8.9323e-01 | |
| 8 | 4.0277e-01 | 3.0341e-01 | 2.3298e-01 | 8.8133e-01 | 8.5489e-01 | 8.5858e-01 |
| 16 | 1.9925e-01 | 9.8125e-02 | 4.9250e-02 | 9.8798e-01 | 8.6095e-01 | 8.6738e-01 |
| 32 | 9.7134e-02 | 2.7271e-02 | 8.9123e-03 | 1.1286e+00 | 9.2875e-01 | 9.9737e-01 |
| 64 | 4.8052e-02 | 7.2024e-03 | 2.1001e-03 | 6.8806e-01 | 6.0206e-01 | 6.6522e-01 |
| 128 | 2.3946e-02 | 1.8968e-03 | 5.9995e-04 | 2.3326e-01 | 1.8737e-01 | 1.1690e-01 |
| 256 | 1.1961e-02 | 5.1132e-04 | 1.8734e-04 | 8.3218e-02 | 4.9127e-02 | 1.0026e-02 |
| 512 | 5.9790e-03 | 1.4392e-04 | 6.1712e-05 | 3.5668e-02 | 1.2443e-02 | 9.8175e-04 |
| 1024 | 2.9892e-03 | 4.2890e-05 | 2.1006e-05 | 1.6989e-02 | 3.1245e-03 | 2.0274e-04 |

TABLE 6.1

The numerical errors $E_1 := |u - u_h|_{\mathcal{H}^1(\mathcal{T}_h)}$, $E_2 := \|\nabla u - G_h u_h\|_0$ and $E_3 := \|\nabla u - RG_H u_h\|_0$ with $\mu = 0$ over \mathcal{T}_m ($m = 4, 8, 16, \dots, 1024$) for $k = 10, 50$.

| m | k=10 | | | k=50 | | |
|------|------------|------------|------------|------------|------------|------------|
| | E_1 | E_2 | E_3 | E_1 | E_2 | E_3 |
| 4 | 7.6897e-01 | 7.1018e-01 | | 9.0817e-01 | 9.0317e-01 | |
| 8 | 4.6583e-01 | 3.8200e-01 | 3.2520e-01 | 8.5422e-01 | 8.5125e-01 | 8.5155e-01 |
| 16 | 2.1661e-01 | 1.3692e-01 | 7.2280e-02 | 9.8422e-01 | 8.6858e-01 | 8.8154e-01 |
| 32 | 9.9911e-02 | 3.8934e-02 | 9.8256e-03 | 1.0561e+00 | 9.1252e-01 | 9.7172e-01 |
| 64 | 4.8407e-02 | 1.0143e-02 | 1.6968e-03 | 9.5362e-01 | 8.7354e-01 | 1.0050e+00 |
| 128 | 2.3987e-02 | 2.5721e-03 | 3.9594e-04 | 3.3755e-01 | 3.0452e-01 | 2.7731e-01 |
| 256 | 1.1965e-02 | 6.4685e-04 | 9.7480e-05 | 1.0473e-01 | 8.0039e-02 | 2.4874e-02 |
| 512 | 5.9791e-03 | 1.6231e-04 | 2.4212e-05 | 3.9086e-02 | 2.0213e-02 | 1.7802e-03 |
| 1024 | 2.9891e-03 | 4.0800e-05 | 6.0608e-06 | 1.7453e-02 | 5.0653e-03 | 1.9414e-04 |

TABLE 6.2

The numerical errors $E_1 := |u - u_h|_{\mathcal{H}^1(\mathcal{T}_h)}$, $E_2 := \|\nabla u - G_h u_h\|_0$ and $E_3 := \|\nabla u - RG_H u_h\|_0$ with $\mu = 1$ over \mathcal{T}_m ($m = 4, 8, 16, \dots, 1024$) for $k = 10, 50$.

| m | k=10 | | | k=50 | | |
|------|------------|------------|------------|------------|------------|------------|
| | E_1 | E_2 | E_3 | E_1 | E_2 | E_3 |
| 4 | 8.1033e-01 | 7.4112e-01 | | 8.9147e-01 | 8.9225e-01 | |
| 8 | 4.8179e-01 | 4.0019e-01 | 3.4405e-01 | 8.5410e-01 | 8.5123e-01 | 8.5150e-01 |
| 16 | 2.1893e-01 | 1.4118e-01 | 7.2934e-02 | 9.8776e-01 | 8.6974e-01 | 8.8355e-01 |
| 32 | 1.0011e-01 | 3.9575e-02 | 9.4564e-03 | 1.0569e+00 | 9.1404e-01 | 9.7362e-01 |
| 64 | 4.8420e-02 | 1.0227e-02 | 1.6651e-03 | 9.5842e-01 | 8.7845e-01 | 1.0097e+00 |
| 128 | 2.3987e-02 | 2.5827e-03 | 3.9469e-04 | 3.3887e-01 | 3.0595e-01 | 2.8075e-01 |
| 256 | 1.1965e-02 | 6.4817e-04 | 9.7438e-05 | 1.0488e-01 | 8.0231e-02 | 2.5034e-02 |
| 512 | 5.9791e-03 | 1.6221e-04 | 2.4213e-05 | 3.9099e-02 | 2.0237e-02 | 1.7813e-03 |
| 1024 | 2.9891e-03 | 3.6859e-05 | 9.2329e-06 | 1.7454e-02 | 5.0672e-03 | 1.9374e-04 |

TABLE 6.3

The numerical errors $E_1 := |u - u_h|_{\mathcal{H}^1(\mathcal{T}_h)}$, $E_2 := \|\nabla u - G_h u_h\|_0$ and $E_3 := \|\nabla u - RG_H u_h\|_0$ with $\mu = 2$ over \mathcal{T}_m ($m = 4, 8, 16, \dots, 1024$) for $k = 10, 50$.

| m | k=10 | | k=60 | | k=120 | |
|------|----------|----------|----------|----------|----------|----------|
| | E_1 | η_h | E_1 | η_h | E_1 | η_h |
| 4 | 6.95e-01 | | 8.35e-01 | | 8.22e-01 | |
| 8 | 4.03e-01 | 3.76e-01 | 9.25e-01 | 2.22e-01 | 8.51e-01 | 3.36e-02 |
| 16 | 1.99e-01 | 1.96e-01 | 9.95e-01 | 5.70e-01 | 9.16e-01 | 2.33e-01 |
| 32 | 9.71e-02 | 9.71e-02 | 1.06e+00 | 4.93e-01 | 9.89e-01 | 5.26e-01 |
| 64 | 4.81e-02 | 4.81e-02 | 9.80e-01 | 3.50e-01 | 1.04e+00 | 4.47e-01 |
| 128 | 2.39e-02 | 2.40e-02 | 3.63e-01 | 2.59e-01 | 1.19e+00 | 3.16e-01 |
| 256 | 1.20e-02 | 1.20e-02 | 1.18e-01 | 1.12e-01 | 6.71e-01 | 3.53e-01 |
| 512 | 5.98e-03 | 5.98e-03 | 4.57e-02 | 4.54e-02 | 1.95e-01 | 1.79e-01 |
| 1024 | 2.99e-03 | 2.99e-03 | 2.08e-02 | 2.08e-02 | 6.05e-02 | 5.98e-02 |

TABLE 6.4

The numerical errors $E_1 := |u - u_h|_{\mathcal{H}^1(\mathcal{T}_h)}$ and η_h with $\mu = 0$ over \mathcal{T}_m ($m = 4, 8, 16, \dots, 1024$) for $k = 10, 60, 120$.

| m | k=10 | | k=60 | | k=120 | |
|------|----------|----------|----------|----------|----------|----------|
| | E_1 | η_h | E_1 | η_h | E_1 | η_h |
| 4 | 7.69e-01 | | 8.42e-01 | | 8.22e-01 | |
| 8 | 4.66e-01 | 3.57e-01 | 8.87e-01 | 5.44e-02 | 8.55e-01 | 8.54e-02 |
| 16 | 2.17e-01 | 1.96e-01 | 9.29e-01 | 4.30e-01 | 8.78e-01 | 3.58e-02 |
| 32 | 9.99e-02 | 9.84e-02 | 1.02e+00 | 4.07e-01 | 9.30e-01 | 3.89e-01 |
| 64 | 4.84e-02 | 4.84e-02 | 1.14e+00 | 3.09e-01 | 1.01e+00 | 3.70e-01 |
| 128 | 2.40e-02 | 2.40e-02 | 5.43e-01 | 3.19e-01 | 1.11e+00 | 2.77e-01 |
| 256 | 1.20e-02 | 1.20e-02 | 1.61e-01 | 1.49e-01 | 9.68e-01 | 3.36e-01 |
| 512 | 5.98e-03 | 5.98e-03 | 5.35e-02 | 5.30e-02 | 3.00e-01 | 2.60e-01 |
| 1024 | 2.99e-03 | 2.99e-03 | 2.19e-02 | 2.19e-02 | 8.35e-02 | 8.24e-02 |

TABLE 6.5

The numerical errors $E_1 := |u - u_h|_{\mathcal{H}^1(\mathcal{T}_h)}$ and η_h with $\mu = 1$ over \mathcal{T}_m ($m = 4, 8, 16, \dots, 1024$) for $k = 10, 60, 120$.

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